Math 166 Solution to questions, Week 8

1. When is the next test?

2. What is your favorite car?
   Probably a '66 GTO.

3. Will we need to prove $n^{1/n}$ converges?
   I cannot say for sure but you will definitely need to know the techniques used to prove it (L'Hospitals Rule, Indeterminant Forms, Continuous Function Theorem). I would say if you cannot prove it then you are missing some fundamental tools for investigating infinite sequences.

4. What countries have you traveled to?
   Canada.

5. Must L'Hospitals rule be used in a limit or integral?
   L'Hospitals Rule can only be applied when you are taking limits (see the solution to Worksheet 8 for more information). This means you cannot use L'Hospitals rule in an integral unless you are using a limit to solve an infinite integral.

6. Was calculus invented or discovered?
   I used to believe that mathematics were discovered. This leads to the conclusion that mathematics is a fundamental part of the universe and all of mathematics exists whether or not we know about it and that mathematics somehow governs how the universe works. Now I lean more towards mathematics as an invention. It is probably the most useful invention in the history of the world (in fact I think Calculus is the single most useful invention in the world) because it allows us to talk so precisely about physics, biology, chemistry, engineering really any science at all. I am not convinced though that mathematics is necessarily true, it might be that it turns out that it is just a really great way to model the universe.

7. What is the highest number in our language?
   My initial thought is a googolplex. To explain this number you need to understand that a googol is a 1 with 100 zeroes after it. A googolplex is a 1 with a googol zero after it. The highest number ever used in a mathematical proof was in a proof by the famous mathematician Ron Graham and is called Graham’s number. I don’t even understand Graham’s number well enough to try to explain it and I am not even sure if it is bigger than a googolplex. Note that you can easily add one to either of these numbers and have a bigger number which we certainly can say in our language.

8. Why a snowflake problem?
   I love this problem, when i first found it I was very fascinated with it and it is still hard for me to understand. I thought it would be intellectually stimulating, although maybe I should not have given you guys so many hints!
9. How do we go from \( \lim_{n \to \infty} \frac{\ln \left( \frac{n+1}{n-1} \right)}{1/n} \) to \( \lim_{n \to \infty} \frac{-2/(n^2-1)}{-1/n^2} \)?

Let’s break this down. First of all notice that the first limit is of the form \( \frac{0}{0} \), this is because the term inside the natural log goes to 1; we can see this by multiplying the numerator and denominator of that term by \( 1/n \) as follows

\[
\lim_{n \to \infty} \frac{n+1}{n-1} = \lim_{n \to \infty} \frac{(n+1)\frac{1}{n}}{(n-1)\frac{1}{n}} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} = \frac{1 + 0}{1 - 0} = 1.
\]

Now we remember that \( \ln(1) = 0 \) and of course \( \lim_{n \to \infty} \frac{1}{n} = 0 \) so we have a form to which we can apply L’Hôpital’s rule, namely \( \frac{0}{0} \).

Now we have to take the derivative of \( \ln \left( \frac{n+1}{n-1} \right) \) and \( \frac{1}{n} \) with respect to \( n \). Recall that to take the derivative of the natural log part we are going to have to use the chain rule. Also recall that in general if there is a function \( f(x) \) inside the natural log function when we differentiate we get \( \frac{d}{dx} \ln[f(x)] = \frac{f'(x)}{f(x)} \) and we will also have to use the quotient rule for the \( \frac{n+1}{n-1} \) part. We will also use the fact that dividing by a fraction is the same as multiplying by its inverse.

\[
\frac{d}{dn} \left[ \ln \left( \frac{n+1}{n-1} \right) \right] = \frac{1}{n+1} \cdot \frac{(n-1) \cdot 1 - (n+1) \cdot 1}{(n-1)^2} = \frac{n-1}{n+1} \cdot \frac{n-1 - n-1}{(n-1)^2} = \frac{-2}{(n+1)(n-1)} = \frac{-2}{n^2 - 1}.
\]

The derivative of \( 1/n \) is easier if we write it as \( n^{-1} \), then \( \frac{d}{dn} [n^{-1}] = -n^{-2} = -\frac{1}{n^2} \). Now when we apply L’Hôpital’s rule we combine these two results to get

\[
\lim_{n \to \infty} \frac{\ln \left( \frac{n+1}{n-1} \right)}{1/n} = \lim_{n \to \infty} \frac{-2/(n^2-1)}{-1/n^2}.
\]

10. What is the definition of diverges and converges?

The technical math definition of a sequence \( a_n \) converging to some limit \( L \) is that given any really small number, we will call this number \( \epsilon \), that is bigger than 0 we can find some number \( N \) such that if \( n \geq N \) then \( |a_n - L| < \epsilon \). The idea is that the sequence will eventually (that is what the \( N \) is for) gets as close to the limit as we want it to get (where \( \epsilon \) tells us how close we want to get) and remains close to the limit. A sequence diverges if it never gets close and stays close to any number.

As an example, the sequence \( a_n = 1 + \frac{1}{n} \) converges to 1. This is true because if you give me some \( \epsilon \) that is really small, say \( \epsilon = \frac{1}{10000000000000000} \), I can find an \( N \) where if \( n \geq N \) then every \( a_n \) is within \( \epsilon \) of the limit 1.

In our case \( N \) can be \( 10000000000000000 \) because every term after \( a_N \) is closer than \( \frac{1}{10000000000000000} \) to 1.

A sequence that diverges doesn’t necessarily have to go to infinity. For example 1, -1, 1, -1, 1, ... diverges because it is always jumping back and forth between -1 and 1 and never stays close to either one of them.

11. Do you like the Fibonacci sequence?

Yeah, it is pretty cool.
12. I know that \( \lim_{n \to \infty} \frac{a^n}{b^n} = \frac{a}{b} \) but what about \( \lim_{n \to \infty} \frac{a^n}{b^n} \)?

Well, that kind of doesn’t make sense. \( a^n \) and \( b^n \) are not really defined, they are best defined by \( \lim_{n \to \infty} a^n \) or \( \lim_{n \to \infty} b^n \). So I will assume that is what you meant. What can we say about

\[
\lim_{n \to \infty} \frac{a^n}{b^n} = ?
\]

Well, to make this easier we can rewrite this as

\[
\lim_{n \to \infty} \left( \frac{a}{b} \right)^n = ?
\]

And we can generalize and instead of thinking of a fraction we can just say that \( x \) is any number and ask the question

\[
\lim_{n \to \infty} x^n = ?
\]

We will start easy, if \( x = 1 \) then \( 1^n = 1 \) for any number \( n \), thus \( \lim_{n \to \infty} 1^n = 1 \). Now, if \( 0 \leq x < 1 \) the solution is presented in problem 6 of Worksheet 8.

If \( x > 1 \) then we will use our logarithm tricks from Worksheet 8. Let \( a_n = x^n \) then \( \ln(a_n) = \ln(x^n) = n \ln(x) \). Since \( x > 1 \) we know that \( \ln(x) \) is a positive number. Now

\[
\lim_{n \to \infty} \ln(a_n) = \lim_{n \to \infty} n \ln(x) = \infty.
\]

This means \( \ln(a_n) \to \infty \) and if we look at the graph of \( \ln(x) \) this means that \( a_n \to \infty \).

13. What does \( n! \) mean?

The exclamation point is the factorial operation, so \( n! \) is read as 'en factorial.' We could define it recursively as \( 0! = 1, n! = n \cdot (n - 1)! \). Examples are better though:

\[
\begin{align*}
0! & = 1 \\
1! & = 1 \\
2! & = 2 \cdot 1 \\
3! & = 3 \cdot 2 \cdot 1 = 6 \\
4! & = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
& \vdots \\
n! & = n(n - 1)(n - 2)(n - 3) \cdots 3 \cdot 2 \cdot 1.
\end{align*}
\]

14. Are there any other cool fractals like the snowflake one?

Yes, if you youtube 'fractal zoom' you will see an awesome video about the Mandelbrot fractal, one of the first fractals ever discovered. The video basically zooms into the fractal and reveals that no matter how deep you go there is always going to be really neat and complexity and structure at the same time. One way to describe fractals is that they are infinitely self-similar.
15. Do you consider Batman a super hero?

Yes, I think he is one of the best super heroes because he is a normal person who just happens to have lots of resources. To me this means that I could be a super hero given the right circumstances which is pretty awesome.

16. What is your favorite thing to do (besides teaching our class, of course)?

I really like spending time with my daughter and cooking with my wife. My favorite non-family activity might be playing foosball, I really like foosball.

17. What came first the chicken or the egg?

I think I answered this already but the egg came first.

18. Do I have to ask a question each week?

Yes, this is how I take attendance which is part of your recitation grade.

19. Solve \[ \lim_{n \to \infty} \frac{1}{n^4} \int_{1}^{n} 3x^2 - \cos^2(x) \, dx. \]

First we will use the limit proper \( \lim_{n \to \infty} f(x)g(x) = \lim_{n \to \infty} f(x) \lim_{n \to \infty} g(x) \) to split this up (and we will recombine it later). We will also use the trig identity \( \cos^2(x) = \frac{1 + \cos(2x)}{2} \) to make the integral easier.

\[
\lim_{n \to \infty} \frac{1}{n^4} \int_{1}^{n} 3x^2 - \cos^2(x) \, dx = \lim_{n \to \infty} \frac{1}{n^4} \cdot \lim_{n \to \infty} \int_{1}^{n} 3x^2 - \frac{1 + \cos(2x)}{2} \, dx = \]

\[
\lim_{n \to \infty} \frac{1}{n^4} \cdot \lim_{n \to \infty} \left[ x^3 - \frac{x}{2} - \frac{\sin(2x)}{4} \right]_{1}^{n} = \lim_{n \to \infty} \frac{1}{n^4} \cdot \lim_{n \to \infty} \left[ n^3 - \frac{n}{2} - \frac{\sin(2n)}{4} - 1^3 + \frac{1}{2} + \frac{\sin(2)}{4} \right] = \]

\[
\lim_{n \to \infty} \frac{1}{n^4} \left[ n^3 - \frac{n}{2} - \frac{\sin(2n)}{4} - 1^3 + \frac{1}{2} + \frac{\sin(2)}{4} \right] = \]

\[
\lim_{n \to \infty} \frac{n^3}{n^4} - \frac{n}{2n^4} - \frac{\sin(2n)}{4n^4} - \frac{1}{n^4} + \frac{1}{2n^4} + \frac{\sin(2)}{4n^4}. \]

All of these terms are zero so our original limit is zero.

20. What is the best way to determine the formula for a sequence?

I would say you write out as many terms as it takes until you see a pattern emerge or try to guess what the next few terms will look like. Don’t try to tackle the whole thing at once, break it down into easier parts. Maybe just figure out the numerator first and then figure out the denominator. There are also several standard ways of writing things, for example if you want to alternate signs you use \((-1)^n\) or \((-1)^{n+1}\) depending on what sign you start with. Odd numbers are written as \(2n + 1\) and even numbers are written as \(2n\). Knowing perfect squares is a good idea too 1, 4, 9, 16, 25, 36, ... maybe even perfect cubes 1, 8, 27, 64, 125, ...
21. What is \( \sum_{k=1}^{\infty} \frac{4^{k-1}}{9^k} \)?

I will answer a more general question which will also solve your problem. Assume \( 0 < r < 1 \) and consider \( \sum_{k=0}^{\infty} ar^k = a + ar + ar^3 + ar^3 + \cdots \). So this is a sum that goes on forever. Let’s call this sum \( S \), in other words we are going to assume that

\[
S = a + ar + ar^3 + ar^3 + \cdots
\]

and notice that if we subtract \( a \) from both sides we get

\[
S - a = ar + ar^3 + ar^3 + \cdots
\]

(keep this in mind). Now we will take the original equation \( S = a + ar + ar^3 + ar^3 + \cdots \) and multiply both sides by \( r \) and get

\[
Sr = ar + ar^3 + ar^3 + \cdots = S - a.
\]

Now we solve for \( S \) and get

\[
S = \frac{a}{1 - r}.
\]

So in our case

\[
S = \sum_{k=1}^{\infty} \frac{4^{k-1}}{9^k} = \sum_{k=1}^{\infty} \frac{4^{k-1}}{4 \cdot 9^k} = \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k = \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k.
\]

Now we have to be very careful, if you want to use the equation

\[
\sum_{k=0}^{\infty} ar^k = a + ar + ar^3 + ar^3 + \cdots = \frac{a}{1 - r}.
\]

Your sum has to start at \( k = 0 \) or we have to change the resulting equation. They are both essentially the same process so I will show you how to take the sum we have \( \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k \) and manipulate it to start at \( k = 0 \) so we can use the nice formula. Let’s just see what these two sums look like:

\[
\sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k = \frac{1}{4} \left( \frac{4}{9} \right)^1 + \frac{1}{4} \left( \frac{4}{9} \right)^2 + \frac{1}{4} \left( \frac{4}{9} \right)^3 + \frac{1}{4} \left( \frac{4}{9} \right)^4 + \cdots
\]

\[
\sum_{k=0}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k = \frac{1}{4} \left( \frac{4}{9} \right)^0 + \frac{1}{4} \left( \frac{4}{9} \right)^1 + \frac{1}{4} \left( \frac{4}{9} \right)^2 + \frac{1}{4} \left( \frac{4}{9} \right)^3 + \frac{1}{4} \left( \frac{4}{9} \right)^4 + \cdots
\]

So they actually only differ by one term, the \( \frac{1}{4} \left( \frac{4}{9} \right)^0 = 1 \) term. So if we subtract \( \frac{1}{4} \) from \( \sum_{k=0}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k \) we will get \( \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k \). Be sure you are paying close attention to where our \( k \) values are starting, otherwise this might not make sense. Now we can write

\[
\sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k - \frac{1}{4} = \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k
\]

and we know from our nice formula that
\[ \sum_{k=0}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k = \frac{1}{4 - 4/9} = \frac{19}{4} \frac{9}{5} = \frac{9}{20}. \]

So finally we can say that

\[ \sum_{k=1}^{\infty} \frac{1}{4} \left( \frac{4}{9} \right)^k = \frac{9}{20} - \frac{1}{4} = \frac{1}{5}. \]