Two techniques that we will be using a lot are L’Hospital’s rule (actually due to the mathematician Bernoulli but L’Hospital was rich and Bernoulli wasn’t) and the continuous function theorem.

L’Hospital’s Rule
Let \( \lim_{x \to c} \), \( \lim_{x \to c^+} \), \( \lim_{x \to c^-} \), \( \lim_{x \to \pm \infty} \), or \( \lim_{x \to 0^+} = \infty \), then if \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \frac{\pm \infty}{\pm \infty} \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \) as long as \( \lim g(x) \neq 0 \).

There are a couple more technicalities but you get the idea.

Continuous Function Theorem
If \( a_n \to L \) ( \( \lim_{n \to \infty} a_n = L \) ) and \( f(x) \) is continuous for all \( a_n \) and \( L \) then \( f(a_n) \to f(L) \) ( \( \lim_{n \to \infty} f(a_n) = f(L) \) ).

1. Determine whether \( (1 + \frac{x}{n})^n \) converges or diverges. If it converges find the value it converges to.
   
   First we will label the sequence \( a_n = \left(1 + \frac{x}{n}\right)^n \). This basically gives us a formula for any term of the sequence, so the first term of the sequence is \( a_1 \) and wherever we see an \( n \) we replace it with a 1, so \( a_1 = \left(1 + \frac{x}{1}\right)^1 \).
   
   Similarly \( a_5 = \left(1 + \frac{x}{5}\right)^5 \). Now, we try to find the limit of the sequence and get
   
   \[
   \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \ln(1 + 0) = 0. 
   \]
   
   which is an in determinant form. Now we use the continuous function theorem, in particular we want to get the exponent down so we use \( \ln(x) \). We get
   
   \[
   \ln(a_n) = \ln \left(\left(1 + \frac{x}{n}\right)^n\right) = n \ln \left(1 + \frac{x}{n}\right). 
   \]
   
   If we take the limit of this we still get an in determinant form
   
   \[
   \lim_{n \to \infty} \ln(a_n) = \lim_{n \to \infty} n \ln \left(1 + \frac{x}{n}\right) = \infty \cdot 0. 
   \]
   
   However, we can rewrite this to get a form that we can apply L’Hospitals rule too, in particular
   
   \[
   \lim_{n \to \infty} n \ln \left(1 + \frac{x}{n}\right) = \lim_{n \to \infty} \frac{\ln \left(\left(1 + \frac{x}{n}\right)^n\right)}{\frac{1}{n}} = 0. 
   \]
   
   Yes! Now we can apply L’Hospitals rule! Note that we are taking derivatives with respect to \( n \) and that when we take the derivative of the top function we have to use the chain rule.
   
   \[
   \lim_{n \to \infty} \frac{\ln \left(\left(1 + \frac{x}{n}\right)^n\right)}{\frac{1}{n}} = \left(1 + \frac{x}{n}\right)^n \lim_{n \to \infty} \frac{n}{1 + x/n} \cdot \frac{-x}{n^2} \cdot \frac{-n^2}{1} = x. 
   \]

Now we are almost done. What we just showed was that \( \ln(a_n) \to x \), but we can use the continuous function theorem again to say \( e^{\ln(a_n)} \to e^x \), but \( e^x \) and \( \ln(x) \) are inverses of each other so \( a_n \to e^x \).
2. We know that \( e^1 \approx 2.718281828459045 \), use question 1 and find which value of \( n \) we need to get within .1 of the actual number.

Since we are looking to approximate \( e^1 \) then we use the sequence \( a_n = \left( 1 + \frac{1}{n} \right)^n \). There are a couple ways to do this (that I know of). You can calculate \( a_1 = 2, a_2 = 2.25, a_3 = 2.370, \ldots \), or you can graph the function \( \left( 1 + \frac{x}{n} \right)^n \) on your graphing calculator and trace the graph until you get to 2.62.

3. Come up with a sequence that converges to \( e^\pi \).
\[ a_n = \left( 1 + \frac{\pi}{n} \right)^n \]

4. Does \( \sqrt[n]{n} \) converge? If so to what value?

Here we use the same tool as in the first problem, we let \( a_n = \sqrt[n]{n} = n^{1/n} \) and notice that \( \lim_{n \to \infty} n^{1/n} = \infty \)
which is indeterminant. Now we take the natural log and get \( \ln(a_n) = \ln(n^{1/n}) = (1/n)\ln(n) = \frac{\ln(n)}{n} \). Now we take the limit and have \( \lim_{n \to \infty} \ln(a_n) = \lim_{n \to \infty} \frac{\ln(n)}{n} = \infty \) so we can use L’Hospital.

\[
\lim_{n \to \infty} \frac{\ln(n)}{n} = L'H = \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0.
\]

But we have a little more work to do, we just showed that \( \ln(a_n) \to 0 \) thus \( e^{\ln(a_n)} \to e^0 \) so \( a_n \to 1 \).

5. Does \( x^{1/n} \) converge \( (x > 0) \)? If so to what value?

Same story, different problem \( a_n = x^{1/n} \). To some of you this may seem obvious but if it is obvious then we should certainly be able to justify it mathematically. Here we are going to take the natural log right away, \( \ln(a_n) = \ln(x^{1/n}) = 1/n \ln(x) = \frac{\ln(x)}{n} \). Now we know that \( x \) is a fixed value so \( \ln(x) \) is also fixed. When we take our limit we have

\[
\lim_{n \to \infty} \ln(a_n) = \lim_{n \to \infty} \frac{\ln(x)}{n} = 0.
\]

So we just showed that \( \ln(a_n) \to 0 \) so \( a_n \to e^0 = 1 \).
6. Does $x^n$ for $0 \leq x < 1$ converge? If so to what value? You will have to know what $\ln(x)$ looks like and use your intuition.

Here is a graph of $\ln |x|$, note that there is a vertical asymptote at $x = 0$. Observe that if $x = 0$ then $x^n = 0$ for all $n$ so $0^n \to 0$. Now if $0 < x < 1$ then $\ln(x)$ is a negative number.

Label the sequence $a_n = x^n$ and take the natural log right away to get $\ln(a_n) = \ln(x^n) = n \ln(x)$. Now we take our limit and get

$$\lim_{n \to \infty} \ln(a_n) = \lim_{n \to \infty} n \ln(x) = \infty \cdot (\text{a negative number}) = -\infty.$$ 

So $\ln(a_n) \to -\infty$ and by looking at our graph this means $a_n \to 0$. The proof for $-1 < x \leq 0$ is a bit more tricky but the graph should give you an idea that $x^n \to 0$ for any $|x| < 1$.

7. Let’s say I am on the 2 yard line of a football field and I want to get to the end zone. If I take a step that is 1 yard long, then another step that is 1/2 a yard long, another step that is 1/4 of a yard long and so on, how many steps do I have to take to reach the endzone?

This might be my favorite proof. Notice that we want to add up how far we have travelled so we sum up

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

where the triple dots mean the sum goes on FOREVER!! So of course if we add up an infinite number of things the sum should also be infinite... get ready for the awesome.

Let $S$ be the sum of all of our terms, that is $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Now multiply both sides of the equation by 2 and the equality will still hold, remember that every term on the right will get multiplied by 2 so we get

$$2S = 2 + \frac{2}{2} + \frac{2}{4} + \frac{2}{8} + \cdots = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$ 

However, on the right side notice that our original sum has shown up again and since we have called that sum $S$ we can substitute it in and get

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 + S.$$ 

Putting this together we have

$$2S = 2 + S.$$ 

Subtract $S$ from both sides and we get $S = 2$, so all of those terms added up equal 2! This means if we start at the 2 yard line we will never reach the end zone because as soon as we stop we will be shy of 2 yards. Super sweet.
8. In a similar scenario, I am on the 5 yard line and I take a 1 yard step, then a 1/2 yard step, then a 1/3 yard step, then a 1/4 yard step and so on. How many steps do I have to take to reach the end zone?

So, thinking about the previous scenario you would think that this is not possible. However, let’s look at the series \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \). Now we are going to compare some of the terms with \( \frac{1}{2} \). Here is what I mean:

\[
1 \geq \frac{1}{2} \\
\frac{1}{2} \geq \frac{1}{2} \\
\frac{1}{3} + \frac{1}{4} = \frac{7}{12} \geq \frac{1}{2}
\]

So we can kind of see a pattern emerging. Now we are going to exploit this pattern (this is what mathematicians do, we exploit patterns). Let’s look at the next 4 terms, \( \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \). Notice that each of the terms is greater than or equal to \( \frac{1}{8} \). This means

\[
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.
\]

If we look at the next 8 terms we will notice that they are each greater than or equal to \( \frac{1}{16} \) so there sum is greater than or equal to \( \frac{8}{16} = \frac{1}{2} \! \). We can do this as many times as we want since the series goes on forever so we know that

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.
\]

Craziness, so we can actually turn around and march to the other endzone!