This worksheet is about the Fundamental theorems of calculus and u-substitution. The ability to analyze a solution is a very important skill and will help you analyze your own work in the future. To give you practice with this I am going to have you analyze some solutions that I came up with grade each one. The problems with be worth 4 points each. 1 point will be awarded for organization, 1 point for showing all work, 1 point for justifying the steps and 1 point for the correct solution.

1. Find $dy/dx$ if $y = \frac{d}{dx} \int_{a}^{x} (t^3 + 1) \, dt$.

   $f(t) = t^3 + 1$ so $dy/dx = x^3 + 1$.

2. Find $dy/dx$ if $y = \frac{d}{dx} \int_{x}^{a} (t^3 + 1) \, dt$.

   $dy/dx = x^3 + 1$.

3. Find $dy/dx$ if $y = \frac{d}{dx} \int_{b}^{x^2+2x-3} (t^3 + 1) \, dt$.

   $dy/dx = (x^3 + 1)(2x + 2)$.

   Because we take the derivative of the top part and multiplied.
4. Find \( \frac{dy}{dx} \) if \( y = \frac{d}{dx} \int_{\sin(x)}^{3} \frac{1}{t} \, dt \).

First we let \( y = G(u) = \int_{u}^{3} \frac{1}{t} \, dt \) and note that \( u(x) = \sin(x) \). To apply the fundamental theorem of calculus we need to switch the bounds on our integral which also changes the sign of the integral so we have

\[
G(u) = -\int_{3}^{u} \frac{1}{t} \, dt.
\]

We are looking for \( \frac{dy}{dx} \) so when we take the derivative of \( y \) we must remember that \( u \) is a function of \( x \) so the chain rule is required.

\[
\frac{d}{dx}[G(u)] = G'(u) \cdot u'(x) = -\ln |u| \cdot \cos(x) = -\ln |\sin(x)| \cdot \cos(x).
\]

5. Suppose that \( f \) has a positive derivative for all values of \( x \) and that \( f(1) = 0 \). Which of the following statements must be true about the function

\[
g(x) = \int_{0}^{x} f(x) \, dx.
\]

Each part is worth 4 points.

(a) \( g \) is a differentiable function of \( x \).

True, it’s obvious.

(b) \( g \) is a continuous function of \( x \).

False, not possible because of \( f \).

(c) The graph has a horizontal tangent at \( x = 1 \).

The function \( f \) is always increasing and \( f(1) = 0 \) means that the graph of \( f \) is below the \( x \)-axis from 0 to 1 and then is above the \( x \)-axis from 1 to \( \infty \). The function \( g(x) \) tells us the area between the graph of \( f \) and the \( x \)-axis from 0 to \( x \). When \( 0 < x < 1 \) \( g(x) \) is going to be negative and since we are adding more area below that graph \( g'(x) \) is also going to be negative. However, when \( x = 1 \) the area under the graph of \( f \) is not changing so \( g'(1) = 0 \). Finally, when \( 1 < x \) we are adding positive area so \( g'(x) \) will be positive.

To sum it up:

\( g'(x) < 0 \) from \( x = 0 \) to \( x = 1 \)
\[ g'(x) = 0 \text{ when } x = 1 \]
\[ g'(x) > 0 \text{ when } x > 1, \]
thus \( g \) has a horizontal tangent line at \( x = 1 \).

(d) \( g \) has a local maximum at \( x = 1 \).
False, the final summary of part (c) tells us this is not the case.

(e) \( g \) has a local minimum at \( x = 1 \).
True.

(f) The graph of \( g \) has an inflection point at \( x = 1 \).
False. We find the second derivative of \( g(x) \) which is \( f''(x) \) which is always positive.

(g) The graph of \( dg/dx \) crosses the \( x \)-axis at \( x = 1 \).
\[ f(1) = 0 \]

6. Evaluate \[ \int_{0}^{\pi} 1 + \cos(x) \, dx. \Rightarrow \pi + 1 - 1 = \pi \]
7. Evaluate \( \int 2(2x + 5)^3 \, dx \).

Let \( u = 2x + 5 \), then \( du = 2 \, dx \). Solving for \( dx \) we have \( dx = \frac{du}{2} \). Now we substitute \( u \) and \( dx \) and pull out the constant 2 and have

\[
\int 2(2x + 5)^3 \, dx = 2 \int u^3 \frac{du}{2} = \int u^3 \, du = \frac{u^4}{4} + c.
\]

Now we substitute \( u \) back in and we have

\[
\int 2(2x + 5)^3 \, dx = \frac{(2x + 2)^4}{2}.
\]

8. Evaluate \( \int x^3 \sqrt{x^2 + 1} \, dx \).

Let \( u = x^2 + 1 \) then \( du = 2x \, dx \). Solve for \( dx \) and we have \( dx = \frac{du}{2x} \). Now substitute and we have

\[
\int x^3 \sqrt{x^2 + 1} \, dx = \int x^3 u^{1/2} \, du = \int x^2 u^{1/2} \, du.
\]

But now our variables are mixed together. We would like to get \( x^2 \) in terms of \( u \)... oh yeah, we know \( u = x^2 + 1 \) so subtract 1 from both sides and we have \( u - 1 = x^2 \). Now we make this substitution and have

\[
\int x^2 u^{1/2} \, du = \int (u - 1)u^{1/2} \, du = \int u^{3/2} - u^{1/2} \, du = \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + c.
\]